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Strong Approximation via Sidon Type Inequalities*

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The aim of this paper is twofold. First we want to show how a duality relation provides a vehicle to deduce strong summability and approximation properties of Fourier series from some basic inequalities, called Sidon type inequalities. This way the technicalities concerning several strong summability and approximation problems can be reduced to proving such inequalities. On the other hand, we will isolate two properties that induce the sharpest version of these inequalities for a number of orthonormal systems, find their counterparts in terms of strong approximation, and show some of their consequences. We note that these results are known to be the best possible for the trigonometric system.

1. INTRODUCTION

The paper consists of four sections. The first one contains the basic concepts and results that are used later. In the second one we prove a duality theorem which plays an essential role in most of the results of the last two sections. The aim of the third section is to show a general theorem about the equivalence of strong summability properties of orthonormal systems and Sidon type inequalities. We also deal with the case of exponentional summability, that turned out to be the best possible for the trigonometric and the Walsh systems, and the corresponding Hardy type Sidon inequality. We show, using the concept of atomic decomposition, that the latter one can be characterized by two simple properties. Similarly to the third section the fourth one starts with a general theorem. Namely, we prove the equivalence of shifted Sidon type inequalities and the rate of convergence of strong oscillations of Fourier series. (The strong oscillation is defined by the generalized de la Vallée Poussin means.) Then again we take the case of Hardy type Sidon inequalities. We identify the basic property of

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strong oscillation that can be considered as the dual of the atomic decomposition. We show that a number of classical results are straightforward consequences of this property. We note that our approach provides a general method for investigating the strong summability and approximation properties of orthonormal systems. Examples are given after each theorem. In particular, the reason behind the difference between the strong summability and approximation properties of the trigonometric and the conjugate trigonometric Fourier series is very clear from our point of view.

A, C, C_1, C_2 will denote positive constants, not necessarily the same in different occurrences, throughout the paper. It will be clear from the text on which quantities they do and on which they do not depend.

$L_p = L_p[0, 1)$ $(1 \leqslant p \leqslant \infty)$ will denote the usual Banach space with the corresponding norm $\| \ \|_p$. We shall denote the set of non-negative integers by $\mathbb{N}$, the set of positive integers by $\mathbb{P}$, and the set of real numbers by $\mathbb{R}$.

Throughout this paper Φ will represent a real orthonormal system with respect to the Lebesgue integral on $[0, 1)$ whose elements are in L_∞. For any $f \in L_1$ let $S^\Phi f$ and $S_n^\Phi f$ $(n \in \mathbb{N})$ denote the Fourier series and the nth partial sum of the Fourier series of f (with respect to Φ). D_n^Φ stands for the nth Dirichlet kernel. Then

$$S_n^\Phi f(x) = \int_0^1 f(t)\, D_n^\Phi(x, t)\, dt \qquad (n \in \mathbb{N}, 0 \leqslant x < 1, f \in L_1).$$

Let $\mathscr{P}_n^\Phi$ $(n \in \mathbb{P})$ denote the set of Φ polynomials of order not greater then n, i.e. the set of linear combinations of the first n elements of Φ. The union of $\mathscr{P}_n^\Phi$'s, i.e., the linear hull of Φ, is called the set of Φ polynomials. It will be denoted by $\mathscr{P}^\Phi$.

$\mathscr{C}^\Phi$ is defined as the closure of $\mathscr{P}^\Phi$ in the norm of L_∞. We note that if Φ is, for instance, the trigonometric system periodic with 1 then $\mathscr{C}^\Phi$ is the set of continuous functions periodic with 1, while if Φ is the Walsh system then $\mathscr{C}^\Phi$ is the space of the so called dyadically continuous functions on $[0, 1)$ (see, e.g., [17]).

Let $\mathscr{L}_n$ stand for the set of dyadic step functions that are constants on the dyadic intervals $[(k-1)2^{-n}, k2^{-n})$ $(0 < k \leqslant 2^n, n \in \mathbb{N})$. The collection of dyadic step functions, i.e., $\bigcup_{n=0}^\infty \mathscr{L}_n$, will be denoted by $\mathscr{L}$. The orthogonal projection $\mathscr{E}_n f$ of L_1 onto $\mathscr{L}_n$ is defined as

$$\mathscr{E}_n f(x) = 2^n \int_{(k-1)2^{-n}}^{k2^{-n}} f \qquad ((k-1)2^{-n} \leqslant x < k2^n,\ 0 \leqslant k \leqslant 2^n,\ n \in \mathbb{N},\ f \in L_1).$$

X will always represent a Banach space with the following properties

(i) $\mathscr{L} \subset X \subset L_1$ and $\|f\|_1 \leqslant \|f\|_X$,

(ii) $\|\mathscr{E}_n f\|_X \leqslant C\, \|f\|_X$ $(n \in \mathbb{N}, f \in X)$.

We note that, for instance, the Orlicz spaces, especially L_p $(1 \leqslant p \leqslant \infty)$, the real non-periodic Hardy space (see, e.g., [11] for definition) and the dyadic Hardy space all satisfy (i) and (ii). The same is true for every dyadic homogeneous Banach space (see, e.g., [17] for definition). Several other examples exist.

Y will always denote the space dual to X. Now we show some immediate consequences of the conditions made for X. First we note that $\mathscr{L} \subset Y$. Indeed, for any $g \in \mathscr{L}$

$$T_g f = \int_0^1 fg \qquad (f \in X)$$

is a bounded linear functional since g is bounded and by $\|f\|_1 \leqslant \|f\|_X$ we have

$$|T_g f| \leqslant \|g\|_\infty \|f\|_1 \leqslant \|g\|_\infty \|f\|_X \qquad (f \in X).$$

Furthermore, using this representation we can define the dual norm on $\mathscr{L}$ as

$$\|g\|_Y = \|T_g\| = \sup_{\|f\|_X \leqslant 1} \int_0^1 fg \qquad (g \in \mathscr{L}). \tag{1}$$

Consequently, we have

$$\left| \int_0^1 fg \right| \leqslant \|f\|_X \|g\|_Y \qquad (f \in X,\ g \in \mathscr{L}). \tag{2}$$

In particular, if $g \in \mathscr{L}_n$ then the Y norm of g can be calculated as follows

$$\|g\|_Y = \sup_{\|f\|_X \leqslant 1,\, f \in \mathscr{L}_n} \int_0^1 fg \qquad (g \in \mathscr{L}_n,\ n \in \mathbb{N}). \tag{3}$$

Indeed, by

$$\|\mathscr{E}_n f\|_X \leqslant C \|f\|_X \qquad \text{and} \qquad \int_0^1 fg = \int_0^1 \mathscr{E}_n f g \qquad (f \in L_1,\ g \in \mathscr{L}_n) \tag{4}$$

we obtain from (1) that

$$\|g\|_Y = \sup_{\|f\|_X \leqslant 1} \int_0^1 \mathscr{E}_n f g \leqslant \sup_{\|f\|_X \leqslant 1,\, f \in \mathscr{L}_n} \int_0^1 fg.$$

The converse inequality is trivial.

We note that, however, we will often use indices 2^n or $j2^n$ in our theorems and their proofs; the extension of the results for arbitrary indices is only a matter of technicalities in most of the cases. Whenever this is not the case we will call attention to that.

2. A DUALITY THEOREM

In this section we show a result that expresses a duality relation. However, such duality could be formalized in a more general way; the form we use will be appropriate for our purpose. Namely, we will use it to transform strong summability, approximation properties of Fourier series, and Sidon type inequalities into each other.

Let $\mathbf{\Delta}$ denote the operator that associates every r-dimensional real vector $(c_k)_{k=1}^r$ $(c_k \in \mathbb{R}, k, r \in \mathbb{N})$ with a dyadic step function as

$$\mathbf{\Delta}(c_k)_{k=1}^r = \sum_{k=1}^{2^n} c_k \chi_{[(k-1)r^{-1}, kr^{-1})},$$

where χ_A denotes the characteristic function of the set $A \subset [0, 1)$.

Let $\Theta_{k,n}(x, \cdot)$ be a Φ polynomial for any $n \in \mathbb{N}$, $1 \leqslant k \leqslant 2^n$, and $0 \leqslant x < 1$. Set

$$T_{k,n} f(x) = \int_0^1 f(t)\, \Theta_{k,n}(x, t)\, dt \qquad (f \in \mathscr{C}^{\Phi}).$$

Then the aforementioned duality relation is the following.

THEOREM 1. *Let $x \in [0, 1)$ and $n \in \mathbb{N}$ be arbitrary but fixed. Then*

$$\|\mathbf{\Delta}(T_{k,n} f(x))_{k=1}^{2^n}\|_Y \leqslant C\, \|f\|_\infty \qquad (f \in \mathscr{C}^{\Phi}) \tag{5}$$

if and only if

$$\frac{1}{2^n} \left\| \sum_{k=1}^{2^n} c_k \Theta_{k,n}(x, \cdot) \right\|_1 \leqslant C\, \|\mathbf{\Delta}(c_k)_{k=1}^{2^n}\|_X \qquad (c_k \in \mathbb{R}, 1 \leqslant k \leqslant 2^n). \tag{6}$$

(*The constant C is the same in* (5) *and* (6).)

We note that the existence of a constant $C > 0$ for which (5) holds follows from the definition of $T_{k,n}$. The same applies to (6). What Theorem 1 really shows is that whenever $C > 0$ is a proper constant for (5) then it is also proper for (6) and vice versa.

3. STRONG SUMMATION

We start this section with a general theorem about the equivalence of strong summability of Fourier series and Sidon type inequalities. First we define the concept of strong summability. Recall that Y is the Banach space dual to X, and that Y contains the dyadic step functions. Then

$$\|\boldsymbol{\Delta}(S_k^{\Phi} f(x)-f(x))_{k=1}^{2^n}\|_Y \qquad (f\in\mathscr{C}^{\Phi},\ n\in\mathbb{N})$$

is called the 2^n*th strong* Y *mean of the Fourier series of* f *at* x. If, for instance, $Y=L_1$ then it is the 2^nth strong Fejér mean, and if $Y=L_p$ $(1\leqslant p<\infty)$ then it is the corresponding strong p-adic mean of Sf, i.e.,

$$\|\boldsymbol{\Delta}(S_k^{\Phi} f(x)-f(x))_{k=1}^{2^n}\|_p=\left(\frac{1}{2^n}\sum_{k=1}^{2^n}|S_k^{\Phi} f(x)-f(x)|^p\right)^{1/p}.$$

We shall say that Φ has the *strong* Y *summability property at* x if for all $f\in\mathscr{C}^{\Phi}$ we have

$$\lim_{n\to\infty}\|\boldsymbol{\Delta}(S_k^{\Phi} f(x)-f(x))_{k=1}^{2^n}\|_Y=0.$$

For this it is obviously necessary that the convergence holds for Φ polynomials, i.e.,

$$\lim_{n\to\infty}\|\boldsymbol{\Delta}(S_k^{\Phi} \varphi(x)-\varphi(x))_{k=1}^{2^n}\|_Y=0 \qquad (\varphi\in\Phi). \tag{7}$$

Remark 1. We note that if the norm of Y satisfies

$$\lim_{n\to\infty}\max_{1\leqslant k\leqslant 2^n}\|\chi_{[(k-1)2^{-n},\,k2^{-n})}\|_Y=0 \qquad (n\in\mathbb{N})$$

then (7) holds no matter what Φ is. For details see [8].

The following theorem is a consequence of Theorem 1.

THEOREM 2. *Let* $x\in[0,1)$. *Then the following two conditions are equivalent.*

(i) *Φ has the strong Y summability property at x.*

(ii) *There exists C such that*

$$\frac{1}{2^n}\int_0^1\left|\sum_{k=1}^{2^n} c_k D_k^{\Phi}(x,t)\right|dt\leqslant C\,\|\boldsymbol{\Delta}(c_k)_{k=1}^{2^n}\|_X \qquad (c_k\in\mathbb{R},\ k,n\in\mathbb{N}) \tag{8}$$

and (7) *holds for* Φ.

Remark 2. Observe that C does not depend on n in (8). In particular, if (8) holds for every $x \in [0, 1)$ with the same C then Φ has the so called *uniform strong Y summability property*. In notation

$$\lim_{n \to \infty} \sup_{0 \leqslant x < 1} \|\mathbf{\Delta}(S_k^{\Phi} f(x) - f(x))_{k=1}^{2^n}\|_Y = 0 \qquad (f \in \mathscr{C}^{\Phi}).$$

Inequalities of the form in (8) are called *Sidon type inequalities with respect to* Φ. So far they have been mainly used to construct integrability and L_1-convergence classes for orthogonal series, especially for trigonometric series. For the history and summaries on Sidon type inequalities we refer to [7, 3].

Example 1. Let $1 \leqslant p < \infty$ be fixed and $1/p + 1/q = 1$. Then we have by Theorem 2 that Φ has the uniform strong L_p summability property

$$\lim_{n \to \infty} \sup_{0 \leqslant x < 1} \frac{1}{n} \sum_{k=1}^{n} |S_k^{\Phi} f(x) - f(x)|^p = 0 \qquad (f \in \mathscr{C}^{\Phi}, n \in \mathbb{P}), \tag{9}$$

if and only if

$$\frac{1}{n} \int_0^1 \left| \sum_{k=1}^{n} c_k D_k^{\Phi}(x, t) \right| dt \leqslant C \left(\frac{1}{n} \sum_{k=1}^{n} |c_k|^q \right)^{1/q} \qquad (0 \leqslant x < 1, c_k \in \mathbb{R}, k, n \in \mathbb{P}). \tag{10}$$

For the trigonometric system, (9) was proved by Hardy and Littlewood [10], and the Sidon type inequality in (10) was proved by Fomin [4] and Bojanic and Stanojević [2] independently. Theorem 2 shows that these results are dual to each other. Concerning strong summation and approximation by trigonometric Fourier series we cite the monograph of Leindler [13] as a general reference.

Now we take the case when X is a Hardy space which is of special interest. Namely, in [14] Schipp pointed out that two properties, that he called *F- and S-properties* (Fejér and Sidon properties), imply a Hardy type Sidon inequality for the corresponding system. Moreover, as it was proved by Fridli in [5, 6], it is the best possible Sidon type inequality in a sense for the trigonometric and Walsh systems. We note that these properties are fundamental inequalities that have several consequences with respect to the convergence, summability, and approximation properties of the orthonormal system. Here we only deal with their impact on strong summability and approximation. For instance we will prove, by using the duality in Theorem 1, that they imply strong summability of exponential order. We also note that in [14] several examples, other then the trigonometric and the Walsh, are given for systems having the F- and S-properties. Among them are the so called *UDMD* and the Ciesielski systems (for definitions see, e.g., [17, 15].)

The F-property is related to the $(C, 1)$ summability of the Fourier series. Namely, Φ is said to have the *F-property at* $x \in [0, 1)$ if

$$\frac{1}{2^n} \int_0^1 \left| \sum_{k=1}^{2^n} D_k^{\Phi}(x, t) \right| dt \leqslant C \qquad (n \in \mathbb{N}). \tag{11}$$

The second one is the shifted version of the original Sidon inequality. We say that Φ has the *shifted S-property at x* if

$$\frac{1}{2^n} \int_0^1 \left| \sum_{k=1}^{2^n} c_{k+\ell} D_{k+\ell}^{\Phi}(x, t) \right| dt \leqslant C \max_{1 \leqslant k \leqslant 2^n} |c_{k+\ell}|, \tag{12}$$

whenever

$$\sum_{k=1}^{2^n} c_{k+\ell} = 0 \qquad (n, \ell \in \mathbb{N}).$$

If such an inequality is required only with $\ell = j2^n$ $(j, n \in \mathbb{N})$ then we say that Φ has the *dyadic shifted S-property at x*.

Remark 3. Clearly, the dyadic shifted S-property is weaker than the shifted S-property. Therefore, whenever it is enough we will only assume that the system has the dyadic shifted S-property. It is easy to check that (11) and (12) together imply (11) for every Fejér kernel, i.e., the index in (11) is not needed to be dyadic power.

Using the above introduced concepts we have the following result about their consequence for strong summability.

THEOREM 3. *Let* $x \in [0, 1)$. *Suppose that* Φ *has the F- and the dyadic shifted S-properties at x. Then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \exp(A\, |S_k^{\Phi} f(x) - f(x)|) - 1 = 0$$

for any $A > 0$ *and* $f \in \mathscr{C}^{\Phi}$.

This result was known for the trigonometric system but was quite new even in that case since it was first proved by Totik in 1980 [17]. From our approach we have that several other systems have the same approximation property. Indeed, recall [14] that the trigonometric, the Walsh, the

Ciesielski, and the *UDMD* systems all have the *F*- and dyadic shifted *S*-properties uniformly in x. Consequently, by Theorem 3 they all have the uniform strong exponential summability property.

In the rest of this section we show an example for a converse application of Theorems 1 and 2. Namely, using the converse Sidon type inequalities proved by Fridli in [5, 6] and Theorem 2 we will prove that Theorem 3 is the best possible for the trigonometric and the Walsh systems.

THEOREM 4. *Let Φ stand for the trigonometric or the Walsh system, and let ψ be a monotonically increasing function defined on $[0, \infty)$ for which $\lim_{u\to 0^+} \psi(u) = 0$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \psi(|S_k^\Phi f(x) - f(x)|) = 0 \qquad (f \in \mathscr{C}^\Phi, 0 \leqslant x < 1) \tag{13}$$

if and only if there exists $A > 0$ such that $\psi(t) \leqslant \exp(At)$ $(0 \leqslant t < \infty)$. Moreover, the convergence is uniform in x.

Remark 4. We note that this result was proved by Totik in [17] for the trigonometric system with the restriction that ψ is continuous.

Notice that the condition $\psi(t) \leqslant \exp(At)$ completely characterizes the functions for which (13) holds. It is quite surprising for the first look since $\exp(A_1 t)$ and $\exp(A_2 t)$ $(A_1 \neq A_2, A_1, A_2 > 0)$ are not of the same order of magnitude. The condition $\psi(t) \leqslant \exp(At)$, however, will come up in a very natural way in our proof. Namely, it comes from strong Y summability when Y is an Orlicz space. The point is that the Young functions $\exp(A_1 t)$ and $\exp(A_2 t)$ generate equivalent Orlicz norms.

4. STRONG APPROXIMATION

In the first part of this section we show the equivalence of shifted Sidon type inequalities and strong approximation properties of Fourier series. We note that the investigation of the problem of strong approximation of trigonometric Fourier series was started by Alexits and Králik [1]. Now we focus our attention on the rate of convergence of strong Y oscillations of Fourier partial sums. The reason is that several well known results, for example, those on the convergence properties of strong de la Vallée Poussin means, can be received as consequences of it.

The *generalized de la Vallée Poussin means* are defined as

$$V_{r,n}^\Phi f = \frac{1}{r} \sum_{k=1}^{r} S_{k+n}^\Phi f \qquad (r, n \in \mathbb{N}, f \in \mathscr{C}^\Phi).$$

Let Y be as before. Then

$$\|\mathbf{\Delta}(S^{\Phi}_{k+n}f(x)-V_{r,n}f(x))_{k=1}^{r}\|_{Y} \qquad (r, n\in\mathbb{N}, 0\leqslant x<1, f\in\mathscr{C}^{\Phi})$$

are called generalized strong Y oscillations of the Fourier series of f. If, for instance, $Y=L_p$ $(1\leqslant p<\infty)$ then they are of the form

$$\left(\frac{1}{r}\sum_{k=1}^{r}|S^{\Phi}_{k+n}f(x)-V_{r,n}f(x)|^{p}\right)^{1/p}.$$

The error of best approximation of f by Φ polynomials of order at most n is defined as follows

$$E^{\Phi}_{n}f=\inf_{p\in\mathscr{P}^{\Phi}_{n}}\|f-p\|_{\infty} \qquad (f\in\mathscr{C}^{\Phi}, n\in\mathbb{P}).$$

The following theorem shows how the rate of convergence of strong oscillations of the Fourier partial sums is connected with shifted Sidon type inequalities.

THEOREM 5. *Let* $0\leqslant x<1$. *Then the following two conditions are equivalent.*

(i) *There exists* C *such that*

$$\frac{1}{2^n}\int_0^1\left|\sum_{k=1}^{2^n}c_k D^{\Phi}_{k+j2^n}(x,t)\right|dt$$

$$\leqslant C\,\|\mathbf{\Delta}(c_k)_{k=1}^{2^n}\|_{X} \qquad \text{with} \quad \sum_{k=1}^{2^n}c_k=0 \quad (j, n\in\mathbb{N}).$$

(ii) *There exists* C *such that*

$$\|\mathbf{\Delta}(S^{\Phi}_{k+j2^n}f(x)-V^{\Phi}_{2^n,j2^n}f(x))_{k=1}^{2^n},\|_{Y}\leqslant CE^{\Phi}_{j2^n}f \qquad (f\in\mathscr{C}^{\Phi}, j, n\in\mathbb{N}).$$

The theorem of course remains true if in both (i) and (ii), $j2^n$ is replaced by $\ell\in\mathbb{N}$.

In order that we can use Theorem 5 to deduce estimations for the generalized strong means, i.e., for

$$\|\mathbf{\Delta}(S^{\Phi}_{j2^n+k}f(x)-f(x))_{k=1}^{2^n}\|_{Y} \qquad (n\in\mathbb{N}),$$

all we need is to estimate $|f(x)-V_{2^n,j2^n}f(x)|$. In other words the only additional information needed is the rate of convergence of generalized de la Vallée Poussin means. It is easy to see that an equivalence similar to Theorem 5 can be established between the rate of convergence of $|f(x)-V_{2^n,j2^n}f(x)|$ and the L_1 norm of the generalized de la Vallé Poussin kernels. We note that the latter one corresponds to the missing case in i) of Theorem 5, namely to the shifted Sidon type inequality when $c_k = 1$ $(k = 1, ..., 2^n)$. The reason why we separated the cases $\sum_{k=1}^{2^n} c_k = 0$, and $c_k = 1$ $(k = 1, ..., 2^n)$ is that the second one is more simple and has a meaning of its own in terms of approximation. Also as it will be seen in Corollary 2 this way we can handle the trigonometric and the conjugate trigonometric Fourier series in the same way. The real reason, however, is the atomic decomposition of Hardy spaces about which more details are given after Corollary 1.

The 2^n*th strong de la Vallée Poussin mean of* f *at* x relative to the Y norm is defined as

$$\|\mathbf{\Delta}(S^{\Phi}_{k+2^n} f(x) - f(x))_{k=1}^{2^n}\|_Y \qquad (n \in \mathbb{N}).$$

In view of the above remarks we have the following corollary of Theorem 5.

COROLLARY 1. *Let* $1 \leqslant p < \infty$ *and suppose that the Sidon type inequality in* (10) *holds for* Φ. *Then for the rate of approximation of the strong de la Vallée Poussin means we have*

$$\left(\frac{1}{n} \sum_{k=1}^{n} |S^{\Phi}_{k+n} f(x) - f(x)|^p\right)^{1/p} \leqslant C E^{\Phi}_n f \qquad (n \in \mathbb{P},\ f \in \mathscr{C}^{\Phi}).$$

Now, similarly to the previous section on strong summability, we will consider the case when Φ satisfies the (dyadic) shifted S property. Recall that it is of particular importance because it leads to exponential summability, the best possible for the trigonometric and Walsh systems. We will show that the shifted S property can be identified with a simple approximation property of the strong oscillation of Fourier series and has several consequences.

Let $\mathscr{H}$ and H denote the *real non-periodic Hardy space* and the *dyadic Hardy space* respectively. For the definitions and the basic properties of these spaces we refer to [11, 17]. Schipp [14] introduced the concepts of the F- and the shifted (or the dyadic shifted) S-properties to prove the Sidon type inequality (8) with $X = \mathscr{H}$ (and with H for the dyadic case). His proof is based on the atomic decomposition of these Hardy spaces. The F property corresponds to the constant 1 atom and the shifted S property corresponds to the other atoms. The advantage of this idea is that the Hardy type Sidon inequalities can be generated from simple basic inequalities.

The dual of $\mathscr{H}$ is essentially the space of functions of *bounded mean oscillation* denoted by $\mathscr{BMO}$. The space $\mathscr{BMO} \subset L_1$ is the collection of functions for which

$$\|f\|_{\mathscr{BMO}} = \left| \int_0^1 f \right| + \sup_{I \subset [0,1)} \frac{1}{|I|} \int_I \left| f - \frac{1}{|I|} \int_I f \right|$$

is finite, where I is an arbitrary subinterval of $[0, 1)$ whose length is denoted by $|I|$. For the dyadic Hardy space H the *dyadic* BMO and its norm can be defined in a similar way with the only modification that I should be a dyadic interval.

If we take $X = H, \mathscr{H}$ then we have by Theorem 5 that the shifted Hardy type Sidon inequalities are equivalent with an approximation property of strong BMO, $\mathscr{B}MO$ oscillations. Recall that by the atomic decomposition of H and $\mathscr{H}$ these Sidon inequalities can be reduced to the shifted S property. Based on the duality in Theorem 5 we identify the corresponding property of strong approximation in the following theorem.

THEOREM 6. *The following three conditions are equivalent.*

(i) *Φ has the uniform shifted S-property.*

(ii) *There exists C such that*

$$\frac{1}{r} \sum_{k=1}^{r} |S_{k+\ell}^{\Phi} f(x) - V_{r,\ell}^{\Phi} f(x)| \leqslant C E_{\ell}^{\Phi} f \qquad (0 \leqslant x < 1, \ell, r \in \mathbb{N}, f \in \mathscr{C}^{\Phi}).$$

(iii) *There exists C such that*

$$\begin{aligned} &\|\mathbf{\Delta}(S_{k+\ell}^{\Phi} f(x) - V_{r,\ell}^{\Phi} f(x))_{k=1}^{r}\|_{\mathscr{BMO}} \\ &\qquad \leqslant C\, E_{\ell}^{\Phi} f \qquad (0 \leqslant x < 1, \ell, r \in \mathbb{N}, f \in \mathscr{C}^{\Phi}). \end{aligned}$$

In the case of the uniform dyadic shifted S-property similar equivalences hold with indices $\ell = j2^n$ $(j, n \in \mathbb{N})$ in (ii) *and in* (iii).

The left side of the inequality in (ii) may be called *generalized strong Fejér oscillation*. Theorem 6 shows that the approximation property of the generalized strong Fejér oscillation in (ii) generates the seemingly stronger result, i.e. the same rate of convergence of the generalized strong $\mathscr{B}MO$ oscillation.

We note that a number of results on strong approximation of Fourier series, many of them classical in the trigonometric case, can be derived from Theorem 6. Here we only show some examples. In order to deduce the following corollary from Theorem 6 all we need are some basic properties of Orlicz norms (see, e.g., [12]) and the well known relation between

Orlicz norms and the BMO, $\mathscr{BMO}$ norms (see, e.g., [9]). The function $\varphi\colon [0, \infty) \mapsto \mathbb{R}$ is called a Young function if it is convex, continuous, $\lim_{t\to 0} \varphi(t)/t = 0$, and $\lim_{t\to\infty} \varphi(t)/t = \infty$.

COROLLARY 2. *Suppose that Φ has the uniform shifted S-property. Then the following inequalities hold uniformly in x.*

(i) *For any $1 \leqslant p < \infty$ we have*

$$\left(\frac{1}{r}\sum_{k=1}^{r} |S^{\Phi}_{k+\ell} f(x) - V^{\Phi}_{r,\ell} f(x)|^p\right)^{1/p} \leqslant CE^{\Phi}_{\ell} f \qquad (0 \leqslant x < 1,\ r, \ell \in \mathbb{N},\ f \in \mathscr{C}^{\Phi}).$$

(ii) *Let φ be a Young function for which $\varphi(u) \leqslant \exp(Au)$ $(u \geqslant 0)$ holds with some $A > 0$. Then there exists $C > 0$ such that*

$$\frac{1}{r}\sum_{k=1}^{r} \varphi(|S^{\Phi}_{k+\ell} f(x) - V^{\Phi}_{r,\ell} f(x)|) \leqslant CE^{\Phi}_{\ell} f$$

$(0 \leqslant x < 1,\ r, \ell \in \mathbb{N},\ f \in \mathscr{C}^{\Phi},\ C\|f\|_{\infty} \leqslant 1)$. *If Φ has the uniform dyadic shifted S-property then the same estimations hold with indices $\ell = j2^n$ and $r = 2^n$* $(j, n \in \mathbb{N})$.

We note that, besides the examples for systems with the *S*-property given before, Theorem 6 and Corollary 2 can also be applied to deduce approximation properties for the conjugate trigonometric series. Indeed, Schipp proved in [14] that the complex trigonometric system satisfies the uniform shifted *S*-property. Therefore (12) holds for the conjugate trigonometric Dirichlet kernels. Consequently, every strong approximation property that follows from the *S* property only will be true not only for the trigonometric but also for the conjugate trigonometric Fourier series. We also note that the *F*-property, i.e., (11), fails to hold for the conjugate kernels. This makes the difference between the approximation properties of the trigonometric and the conjugate trigonometric Fourier series.

EXAMPLE 2. Let $\widetilde{S}_k f$ $(k \in \mathbb{P})$ denote the kth partial sum of the conjugate trigonometric Fourier series and let $\widetilde{V}_{k,n} f$ denote the conjugate generalized de la Vallée Poussin means of $f \in C[0, 1)$ $(k, n \in \mathbb{N})$. We have by Theorem 6 and Corollary 2 that for any $A > 0$ there exists $C > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n} \exp(A\,|\widetilde{S}_{k+n} f(x) - \widetilde{V}_n f(x)|) - 1 \leqslant CE_n f \qquad (C\|f\|_{\infty} \leqslant 1,\ 0 \leqslant x < 1,\ n \in \mathbb{P}).$$

($E_n f$ denotes the error of best approximation by trigonometric polynomials of order not greater then n, and $\tilde{V}_n f(x)$ denotes the nth de la Vallée Poussin mean of the conjugate trigonometric Fourier series.)

In particular, since $f \in \mathrm{Lip}\,\alpha$ $(0<\alpha<1)$ implies $\|\tilde{S}_k f - \tilde{f}\|_\infty \leqslant Ck^{-\alpha}$ $(k \in \mathbb{P})$ we obtain that for any $1 \leqslant p < \infty$ there exists $C>0$ such that

$$\left(\frac{1}{n}\sum_{k=1}^{n} |\tilde{S}_k f(x) - \tilde{f}(x)|^p\right)^{1/p} \leqslant Cn^{-\alpha} \qquad (0 \leqslant x < 1,\, n \in \mathbb{P},\, f \in C[0,1)).$$

Similarly, for any $A>0$ there exists $C>0$ such that

$$\frac{1}{n}\sum_{k=1}^{n} \exp(A\,|\tilde{S}_k f(x) - \tilde{f}(x)|) - 1 \leqslant Cn^{-\alpha} \qquad (0 \leqslant x < 1,\, n \in \mathbb{P},\, C\|f\|_\infty \leqslant 1).$$

Let us suppose that Φ has also the uniform F-property. Thus the de la Vallée Poussin means of the Fourier series converge to the corresponding function. Taking $r = \ell$ in Theorem 6 and Corollary 2 we could easily deduct estimations for the rate of convergence of strong de la Vallée Poussin means. Instead, we will show that an even more general result, that contains the strong de la Vallée Poussin means as special cases, can be derived from the F- and S-properties. Namely, we have the following theorem a trigonometric version of which was proved in [18].

THEOREM 7. *Suppose that Φ has the uniform F- and the uniform dyadic shifted S-properties. Then there exists $C>0$ such that*

(i) $$\|\mathbf{\Delta}(S^{\Phi}_{k_j} f(x) - f(x))_{k=1}^{r}\|_{L_N} \leqslant C\left(\log\frac{2n}{r}\right) E^{\Phi}_{k_1} f \qquad (0 \leqslant x < 1,\, r, n \in \mathbb{P},\, f \in \mathscr{C}^{\Phi}),$$

where $\|\ \|_{L_N}$ is the Orlicz norm generated by the Young function N for which $N(u) = \exp u - 1$ for u great enough, and $0 < k_1 < \cdots < k_r \leqslant n$.

(ii) *Especially, for any $1 \leqslant p < \infty$*

$$\left(\frac{1}{r}\sum_{j=1}^{r} |S^{\Phi}_{k_j} f(x) - f(x)|^p\right)^{1/p} \leqslant C\left(\log\frac{2n}{r}\right) E^{\Phi}_{k_1} f \qquad (0 \leqslant x < 1,\, r, n \in \mathbb{P},\, f \in \mathscr{C}^{\Phi})$$

$(0 < k_1 < \cdots < k_r \leqslant n)$.

Finally, we mention a consequence with respect to the strong de la Vallée means of the above theorem.

COROLLARY 3. *Suppose that Φ has the uniform F- and the uniform dyadic shifted S-properties. Let φ: $[0, \infty) \mapsto \mathbb{R}$ be a monotonically increasing continuous function with $\lim_{u \to 0^+} \varphi(u) = 0$ for which there exists A such that*

$$\varphi(u) \leqslant \exp(Au) \qquad (u \geqslant 0),$$

and

$$\varphi(2u) \leqslant A\varphi(u) \qquad (0 < u < 1).$$

Then

$$\frac{1}{n} \sum_{k=n+1}^{2n} \varphi(|S_k^{\Phi} f(x) - f(x)|) \leqslant C\varphi(E_n^{\Phi} f) \qquad (0 \leqslant x < 1, n \in \mathbb{P}, f \in \mathscr{C}^{\Phi}).$$

We can infer Corollary 3 from part ii) in Theorem 7 by following the proof of the trigonometric version given by Totik in [19].

5. PROOFS

Proof of Theorem 1. First suppose that

$$\|\mathbf{\Delta}(T_{k,n}f(x))_{k=1}^{2^n}\|_Y \leqslant C \|f\|_{\infty} \qquad (f \in \mathscr{C}^{\Phi}, n \in \mathbb{P}).$$

Then for any real c_k's $(1 \leqslant k \leqslant 2^n)$ we have by the duality argument and by (2) that

$$\begin{aligned}
\frac{1}{2^n} \int_0^1 \left| \sum_{k=1}^{2^n} c_k \Theta_{k,n}(x, t) \right| dt &= \sup_{\|f\|_{\infty} \leqslant 1} \frac{1}{2^n} \sum_{k=1}^{2^n} c_k \int_0^1 f(t)\, \Theta_{k,n}(x, t)\, dt \\
&= \sup_{\|f\|_{\infty} \leqslant 1} \frac{1}{2^n} \sum_{k=1}^{2^n} c_k T_{k,n} f(x) \\
&= \sup_{\|f\|_{\infty} \leqslant 1} \int_0^1 (\mathbf{\Delta}(c_k)_{k=1}^{2^n})(\mathbf{\Delta}(T_{k,n}f(x))_{k=1}^{2^n}) \\
&\leqslant \sup_{\|f\|_{\infty} \leqslant 1} \|\mathbf{\Delta}(c_k)_{k=1}^{2^n}\|_X \|\mathbf{\Delta}(T_{k,n}f(x))_{k=1}^{2^n}\|_Y \\
&\leqslant C \|\mathbf{\Delta}(c_k)_{k=1}^{2^n}\|_X. \qquad (14)
\end{aligned}$$

For the proof of the converse direction suppose that

$$\frac{1}{2^n}\left\|\sum_{k=1}^{2^n} c_k \Theta_{k,n}(x,\cdot)\right\|_1 \leqslant C\,\|\boldsymbol{\Delta}(c_k)_{k=1}^{2^n}\|_X$$

holds for any $c_k \in \mathbb{R}$ $(1 \leqslant k \leqslant 2^n,\ k, n \in \mathbb{P})$.

Clearly, any function in $\mathscr{L}_n$ can be written in the form $\boldsymbol{\Delta}(c_k)_{k=1}^{2^n}$ with the proper choice of the c_k's. Therefore, we have by (3) that

$$\begin{aligned}
\|\boldsymbol{\Delta}(T_{k,n}f(x))_{k=1}^{2^n}\|_Y &= \sup_{\|\boldsymbol{\Delta}(c_k)_{k=1}^{2^n}\|_X \leqslant 1} \int_0^1 (\boldsymbol{\Delta}(T_{k,n}f(x))_{k=1}^{2^n})(\boldsymbol{\Delta}(c_k)_{k=1}^{2^n}) \\
&= \sup_{\|\boldsymbol{\Delta}(c_k)_{k=1}^{2^n}\|_X \leqslant 1} \frac{1}{2^n}\sum_{k=1}^{2^n} c_k T_{k,n} f(x) \\
&= \sup_{\|\boldsymbol{\Delta}(c_k)_{k=1}^{2^n}\|_X \leqslant 1} \int_0^1 \frac{1}{2^n}\sum_{k=1}^{2^n} c_k \Theta_{k,n}(x,t)\, f(t)\, dt \\
&\leqslant \|f\|_\infty \sup_{\|\boldsymbol{\Delta}(c_k)_{k=1}^{2^n}\|_X \leqslant 1} \frac{1}{2^n}\left\|\sum_{k=1}^{2^n} c_k \Theta_{k,n}(x,\cdot)\right\|_1 \leqslant C\,\|f\|_\infty .
\end{aligned} \tag{15}$$

The proof of Theorem 1 is complete. ∎

Proof of Theorem 2. Since

$$\mathscr{C}^\Phi \ni f \to \boldsymbol{\Delta}(S_k^\Phi f(x) - f(x))_{k=1}^{2^n} \in Y \qquad (n \in \mathbb{N})$$

is a sequence of linear operators we have by the Banach–Steinhaus theorem that Φ has the strong Y summability property at x if and only if (7) holds and these operators are uniformly bounded. Clearly, the uniform boundedness is equivalent to (5) with $T_{k,n} = S_k^\Phi$ $(1 \leqslant k \leqslant 2^n,\ k, n \in \mathbb{N})$ and with $C > 0$ independent of n. Then by Theorem 1 it is also equivalent to (6), which is identical to (8) in this case. ∎

Proof of Theorem 3. Suppose that Φ has the F- and S-properties at an $x \in [0, 1)$. Then as Schipp proved in [14] the following Sidon type inequality holds true for Φ

$$\frac{1}{2^n}\int_0^1 \left|\sum_{k=1}^{2^n} c_k D_k^\Phi(x,t)\right| dt \leqslant \|\boldsymbol{\Delta}(c_k)_{k=1}^{2^n}\|_H \qquad (c_k \in \mathbb{R},\ k, n \in \mathbb{N}).$$

Recall that H denotes the dyadic Hardy space. Let now M be a Young function for which $M(x) = x \log x$ for large values of x and let the Orlicz space generated by M be denoted by L_M. For the definition and properties of Orlicz spaces we refer to [12]. Now we make use of the following relation between the dyadic Hardy norm and the Orlicz norm $\| \ \|_{L_M}$ (see, e.g., [16])

$$\|\mathbf{\Delta}(c_k)_{k=1}^{2^n}\|_H \leqslant C\, \|\mathbf{\Delta}(c_k)_{k=1}^{2^n}\|_{L_M} \qquad (c_k \in \mathbb{R},\, k,\, n \in \mathbb{N}).$$

Consequently, in the Sidon type inequality above H can be replaced by L_M. It is known (see, e.g., [12]) that the dual of L_M is the Orlicz space generated by the Young function $N(x) = \exp x - 1$ $(x \geqslant 0)$. Also, it is easy to check that the Orlicz norms have the property defined in Remark 1 and so (7) holds for them. Then we have by Theorem 3 that

$$\lim_{n\to\infty} \|\mathbf{\Delta}(S_k^{\Phi} f(x) - f(x))_{k=1}^{2^n}\|_{L_N} = 0 \qquad (f \in \mathscr{C}^{\Phi}). \tag{16}$$

Using the concept of equivalent Young functions (see [12]) we have that (16) holds for every Young function of the form $N(x) = \exp(Ax) - 1$ $(A > 0)$. Since the convergence in Orlicz norm implies the convergence in mean, that is

$$\lim_{n\to\infty} \|f_n - f\|_{L_N} = 0 \qquad \text{implies} \qquad \lim_{n\to\infty} \int_0^1 N(|f_n(t) - f(t)|)\, dt = 0,$$

we can conclude that

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{2^n} \exp(A\, |S_k^{\Phi} f(x) - f(x)|) - 1 = 0$$

for any $A > 0$ and $f \in \mathscr{C}^{\Phi}$. ∎

Proof of Theorem 4. Let ψ be a monotonically increasing function for which (13) and $\lim_{u\to 0^+} \psi(u) = 0$ hold. The "if" part of the statement is a straightforward consequence of Theorem 3.

For the proof of the other part first we suppose that ψ is continuous and convex, i.e., ψ is a Young function. Then in terms of Orlicz spaces (13) means that $\mathbf{\Delta}(S_k^{\Phi} f(x) - f(x))_{k=1}^{2^n}$ tends to 0 in mean as $n \to \infty$. It is known that, however, the convergence in mean does not imply the convergence in the corresponding Orlicz norm in general but it does imply the boundedness. Consequently

$$\sup_{n\in\mathbb{N}} \|\mathbf{\Delta}(S_k^{\Phi} f(x))_{k=1}^{2^n}\|_{L_\psi} < \infty \qquad (f \in \mathscr{C}^{\Phi}).$$

Then we have by the Banach–Steinhaus theorem that the sequence of the operators

$$\mathscr{C}^{\Phi} \ni f \to \mathbf{\Delta}(S_k^{\Phi} f(x))_{k=1}^{2^n} \in L_\psi \qquad (n \in \mathbb{N})$$

is uniformly bounded, i.e.,

$$\|\mathbf{\Delta}(S_k^{\Phi} f(x))_{k=1}^{2^n}\|_{L_\psi} \leqslant C\|f\|_\infty \qquad (f \in \mathscr{C}^{\Phi}, n \in \mathbb{N}).$$

If ϖ stands for the conjugate Young function of ψ then we can deduce by Theorem 1 that the following Sidon type inequality holds true

$$\frac{1}{2^n}\int_0^1 \left|\sum_{k=1}^{2^n} c_k D_k^{\Phi}(x,t)\right| dt \leqslant C\,\|\mathbf{\Delta}(c_k)_{k=1}^{2^n}\|_{L_\varpi} \qquad (c_k \in \mathbb{R}, n \in \mathbb{N}).$$

In [5, 6] Fridli proved the following converse Sidon type inequality

$$\max_{p \in \Pi_n} \frac{1}{2^n}\int_0^1 \left|\sum_{k=1}^{2^n} c_{p_k} D_k^{\Phi}(x,t)\right| dt \geqslant C\,\|\mathbf{\Delta}(c_k)_{k=1}^{2^n}\|_{L_M} \qquad (c_k \in \mathbb{R}, n \in \mathbb{N}),$$

where $M(x) = x \log x$ for large values of x and Π_n denotes the set of permutations of the set $\{1, ..., 2^n\}$ $(n \in \mathbb{N})$. Since the Orlicz norms are rearrangement invariant we have from this result that

$$\|h\|_{L_M} \leqslant C\,\|h\|_{L_\varpi} \tag{17}$$

for every dyadic step function h. M satisfies the so called $\mathbf{\Delta}_2$ condition, i.e., $M(2x) \leqslant CM(x)$ for large values of x. Therefore, the set of dyadic step functions is dense in L_M. Consequently, (17) holds for every $f \in L_M$. This implies the converse relation between the dual norms, i.e.,

$$\|f\|_{L_\psi} \leqslant C\,\|f\|_{L_N} \qquad (f \in L_\psi),$$

where $N(x) = \exp x - 1$ $(x \leqslant 0)$. It is known that this can only be true (see, e.g., [12]) if $\psi \preccurlyeq N$. The partial ordering $\preccurlyeq$ in the set of Young functions is defined as follows. For two Young functions M_1 and M_2 we have $M_1 \preccurlyeq M_2$ if and only if there exist $A > 0$ and $u_0 > 0$ such that $M_1(u) \leqslant M_2(Au)$ $(u > u_0)$.

Applying it to our case we can conclude that there exists $A > 0$ such that

$$\psi(u) < \exp(Au) \qquad (x \geqslant 0),$$

i.e., we proved the theorem in case when ψ is a Young function.

In order to finish the proof let us suppose that ψ is a monotonically increasing function with $\lim_{u \to 0^+} \psi(u) = 0$ that does not satisfy the condition of the theorem. Namely, we suppose that for any $A > 0$ there exists $s > 0$ for which $\psi(s) > \exp(As)$ holds. Let us take the integral function of $\sqrt{\psi}$

$$\vartheta(u) = \int_0^u \sqrt{\psi(t)}\,dt \qquad (u \geqslant 0).$$

Thus ϑ is a Young function and it follows from the definition that

$$\vartheta(u) \leqslant u\sqrt{\psi(u)}, \qquad \vartheta(u) \geqslant \frac{u}{2}\sqrt{\psi\left(\frac{u}{2}\right)}\ (u \geqslant 0). \tag{18}$$

Since for any $A > 0$ there exists $s > 0$ such that $\psi(s) > \exp(As)$, where we may suppose that $s > 2$, we have by the second inequality in (18) that $\vartheta(2s) > \exp((A/2)s)$. That is, ϑ does not satisfy the condition which was proved to be necessary to (13) for Young functions. Therefore, there exists $f \in \mathscr{C}^{\Phi}$ such that

$$\limsup_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{2^n} \vartheta(|S_k^{\Phi} f(x) - f(x)|) > 0. \tag{19}$$

By Cauchy's inequality and by the first part of (18) we have that

$$\frac{1}{2^n}\sum_{k=1}^{2^n} \vartheta(|S_k^{\Phi} f(x) - f(x)|)$$
$$\leqslant \left(\frac{1}{2^n}\sum_{k=1}^{2^n} |S_k^{\Phi} f(x) - f(x)|^2\right)^{1/2} \left(\frac{1}{2^n}\sum_{k=1}^{2^n} \psi(|S_k^{\Phi} f(x) - f(x)|)\right)^{1/2}.$$

The first factor of the right side tends to 0 as $n \to \infty$ since both the trigonometric and the Walsh systems have the uniform strong Y summability property with $Y = L_2$. Consequently, by (19) we have

$$\limsup_{n\to\infty} \frac{1}{2^n}\sum_{k=1}^{2^n} \psi(|S_k^{\Phi} f(x) - f(x)|) = \infty.$$

The proof of the theorem is complete. ∎

Proof of Theorem 5. We can follow the proof of Theorem 1. Namely, let $0 \leqslant x < 1$ and $j, n \in \mathbb{N}$. Set $\Theta_{k,n} = D^{\Phi}_{j2^n+k}$ $(1 \leqslant k \leqslant 2^n)$. First we suppose that (ii) holds, i.e.,

$$\|\mathbf{\Delta}(S^{\Phi}_{j2^n+k} f(x) - V^{\Phi}_{2^n, j2^n} f(x))_{k=1}^{2^n}\|_Y \leqslant C E^{\Phi}_{j2^n} f \qquad (f \in \mathscr{C}^{\Phi}).$$

Then (i) follows by (14) and by considering that in this case

$$\sum_{k=1}^{2^n} c_k T_{k,n} f(x) = \sum_{k=1}^{2^n} c_k (S^{\Phi}_{j2^n+k} f(x) - V^{\Phi}_{2^n, j2^n} f(x))$$

whenever $\sum_{k=1}^{2^n} c_k = 0$.

For the proof of the other direction observe that

$$\int_0^1 \mathbf{\Delta}(S^{\Phi}_{j2^n+k} f(x) - V^{\Phi}_{2^n, j2^n} f(x))_{k=1}^{2^n} = 0 \qquad (f \in \mathscr{C}^{\Phi})$$

and

$$\left\| \mathbf{\Delta} \left(c_k - 2^{-n} \sum_{j=1}^{2^n} c_j \right)_{k=1}^{2^n} \right\|_X \leqslant \|\mathbf{\Delta}(c_k)_{k=1}^{2^n}\|_X + \|\mathscr{E}_0(\mathbf{\Delta}(c_k)_{k=1}^{2^n})\|_X$$

$$\leqslant 2\, \|\mathbf{\Delta}(c_k)_{k=1}^{2^n}\|_X \qquad (c_k \in \mathbb{R},\ 1 \leqslant k \leqslant 2^n).$$

Then similarly to (15) we obtain

$$\|\mathbf{\Delta}(S^{\Phi}_{j2^n+k} f(x) - V^{\Phi}_{2^n,\, j2^n} f(x))_{k=1}^{2^n}\|_Y$$

$$\leqslant 2 \sup_{\|\mathbf{\Delta}(c_k)_{k=1}^{2^n}\|_X \leqslant 1} \frac{1}{2^n} \sum_{k=1}^{2^n} c_k (S^{\Phi}_{j2^n+k}(f-g)(x) - V^{\Phi}_{2^n,\, j2^n}(f-g)(x))$$

$$(g \in \mathscr{P}^{\Phi}_{j2^n}),$$

where $\sum_{k=1}^{2^n} c_k = 0$. Thus the last sum reduces to

$$\sum_{k=1}^{2^n} c_k S^{\Phi}_{j2^n+k}(f-g)(x)$$

and we can finish the proof as for Theorem 1. ▮

Proof of Theorem 6. The proof will be presented for the dyadic case. Namely, we suppose that $\ell = j2^n$ and $r = 2^n$ for some $j, n \in \mathbb{N}$. Recall that the F- and the dyadic shifted S-properties imply the Sidon type inequality in (8) with $X = H$. The proof (see [14]) is based on the concept of atomic decomposition for the dyadic Hardy space H. In term of atoms the F-property corresponds to the constant 1 atom. Excluding it and using the same idea as in [14] we deduce that the uniform dyadic shifted S-property is equivalent to (i) in Theorem 6 uniformly in x. Then it follows from the duality between H and BMO, and from Theorem 5 that (i) is equivalent to (iii).

By definition we have that $\|h\|_1 \leqslant \|h\|_{BMO}$ ($h \in$ BMO). Consequently, (iii) implies (ii). On the other hand, if (ii) holds then using the definition of the BMO norm again we obtain

$$\|\mathbf{\Delta}(S^{\Phi}_{j2^n+k} f(x) - V^{\Phi}_{2^n,\, j2^n} f(x))_{k=1}^{2^n}\|_{BMO}$$

$$= \max_{j2^n \leqslant i2^s < (j+1)2^n} \frac{1}{2^s} \sum_{k=1}^{2^s} |S^{\Phi}_{i2^s+k} f(x) - V^{\Phi}_{2^s,\, i2^s} f(x)|$$

$$\leqslant C \max_{j2^n \leqslant i2^s < (j+1)2^n} E^{\Phi}_{i2^s} f = C E^{\Phi}_{j2^n} f \qquad (j, n \in \mathbb{N},\ f \in \mathscr{C}^{\Phi}),$$

i.e., (ii) implies (iii).

We note that for the proof of the case of the shifted S-property one has to take the corresponding properties of $\mathscr{H}$ and $\mathscr{BMO}$ and to use the extended version of Theorem 5 to arbitrary indices. ∎

Proof of Theorem 7. Again we will use the F- and S-properties to conclude

$$\frac{1}{2^{\ell}}\int_0^1\left|\sum_{k=1}^{2^{\ell}} d_k D_k^{\Phi}(x,t)\right| dt \leqslant C\,\|\mathbf{\Delta}(d_k)_{k=1}^{2^{\ell}}\|_H \qquad (0\leqslant x<1,\ d_k\in\mathbb{R},\ k,\ell\in\mathbb{P}).$$

Recall that $\|f\|_H \leqslant C\|f\|_{L_M}$ $(f\in L_M)$ where $M(x)=x\log x$ for x great enough. On the other hand (see, e.g., [5])

$$\|f\|_{L_M}\approx\int_0^1 |f|\left(1+\log^+\frac{|f|}{\|f\|_1}\right) \qquad (f\in L_M,\ f\not\equiv 0).$$

Consequently,

$$\int_0^1\left|\sum_{k=1}^{2^{\ell}} d_k D_k^{\Phi}(x,t)\right| dt \leqslant C\sum_{k=1}^{2^{\ell}} |d_k|\left(1+\log^+\frac{|d_k|}{2^{-\ell}\sum_{j=1}^{2^{\ell}}|d_j|}\right)$$

$(0\leqslant x<1,\ d_k\in\mathbb{R},\ k,\ell\in\mathbb{P})$.

Let $0<k_1<\ldots<k_r\leqslant n$ $(r,n\in\mathbb{P})$. Define N and K by $2^{N-1}<n\leqslant 2^N$ and $2^{K-1}<r\leqslant 2^K$. Set

$$\Theta_{j,K}=\begin{cases} D_{k_j}^{\Phi} & j=1,\ldots,r\\ 0, & j=r+1,\ldots,2^K.\end{cases}$$

Thus

$$\begin{aligned}
&\frac{1}{2^K}\int_0^1\left|\sum_{j=1}^{2^K} c_j\Theta_{j,K}(x,t)\right| dt\\
&\quad=\frac{1}{2^K}\int_0^1\left|\sum_{j=1}^{r} c_j D_{k_j}(x,t)\right| dt\\
&\quad\leqslant C\frac{1}{2^K}\sum_{j=1}^{r}|c_j|\left(1+\log^+\frac{|c_j|}{2^{-N}\sum_{j=1}^{r}|c_j|}\right)\\
&\quad\leqslant C\frac{1}{2^K}\left(\sum_{j=1}^{2^K}|c_j|\left(1+\log\frac{2^N}{2^K}\right)+\sum_{j=1}^{2^K}|c_j|\left(1+\log^+\frac{|c_j|}{2^{-K}\sum_{\ell=1}^{2^K}|c_\ell|}\right)\right)\\
&\quad\leqslant C\log\frac{2n}{r}\,\|\mathbf{\Delta}(c_j)_{j=1}^{2^K}\|_{L_M}.
\end{aligned}$$

Then we have by Theorem 1 that

$$\left\| \sum_{j=1}^{r} (S_{k_j}^{\Phi} f(x) - g(x)) \chi_{[(j-1)2^{-K}, j2^{-K})} \right\|_{L_N} \leqslant C \log \frac{2n}{r} \|f - g\|_\infty \qquad (g \in P_{k_1}^{\Phi}),$$

where $N(u) = \exp u - 1$ for u great enough.

The proof can be completed by using simple properties of Orlicz norms. ▮

REFERENCES

1. G. Alexits and D. Králik, Über den Annäherungsgrad der Approximation im starken Sinne von stetigen Funktionen, *Magyar Tud. Akad. Mat. Kut. Int. Közl.* **8** (1963), 317–327.
2. R. Bojanic and Č. V. Stanojević, A class of L^1 convergence, *Trans. Amer. Math. Soc.* **269** (1982), 677–683.
3. M. Buntinas and N. Tanović-Miller, New integrability and L^1 convergence classes for even trigonometric series, II, *in* "Approximation Theory, Kecskemét, Hungary," Colloq. Math. Soc. János Bolyai, North-Holland, Amsterdam.
4. G. A. Fomin, A class of trigonometric series, *Mat. Zametki* **14** (1973), 213–222. [In Russian]
5. S. Fridli, An inverse Sidon type inequality, *Studia Math.* **105**, No. 3 (1993), 283–308.
6. S. Fridli, An inverse Sidon type inequality for the Walsh system, *J. Math. Anal. Appl.* **193** (1995), 715–736.
7. S. Fridli, Integrability and L^1-convergence of trigonometric and Walsh series, *Ann. Univ. Sci. Budapest Eötvös Sect. Comput.* **16** (1996), 149–172.
8. S. Fridli and F. Schipp, Strong summability and Sidon type inequalities, *Acta Sci. Math. (Szeged)* **60** (1995), 277–289.
9. A. M. Garsia, "Martingale Inequalities," Seminar Notes on Recent Progress. Math. Lecture Notes Series, Benjamin, New York, 1973.
10. G. H. Hardy and J. E. Littlewood, Sur la série de Fourier d'une fonction à carré sommable, *C. R. Paris* **156** (1913), 1307–1309.
11. B. S. Kashin and A. A. Saakyan, "Orthogonal Series," Transl. Math. Monographs, Vol. 75, Amer. Math. Soc., Providence, 1989.
12. M. A. Krasnosel'skiĭ and M. Rutickiĭ, "Convex functions and Orlicz Spaces," Noordhoff, Groningen, 1961.
13. L. Leindler, "Strong Approximation by Fourier Series," Akad. Kiadó, Budapest, 1985.
14. F. Schipp, Sidon-type inequalities, *in* "Approx. Theory," Lecture Notes in Pure and Appl. Math., pp. 421–436, Vol. 138, Dekker, New York/Basel/Hong Kong, 1992.
15. F. Schipp and W. R. Wade, Norm convergence and summability of Fourier series with respect to certain product systems, *in* "Approx. Theory," Lecture Notes in Pure and Appl. Math., pp. 437–452, Vol. 138, Dekker, New York/Basel/Hong Kong, 1992.
16. F. Schipp, W. R. Wade, and P. Simon (with assistance from J. Pál), "Walsh Series," Hilger, Bristol/New York, 1990.

17. V. Totik, On the generalization of Fejér's summation theorem, *in* "Functions, Series, Operators," Colloq. Math. Soc. János Bolyai, Vol. 35, pp. 1195–1199, North-Holland, Amsterdam, 1980.
18. V. Totik, On the strong approximation of Fourier series, *Acta Math. Sci. Hungar (Szeged)* **35** (1980), 151–172.
19. V. Totik, Notes on Fourier series: Strong approximation, *J. Approx. Theory* **43** (1985), 105–111.